Optimal Three-Dimensional Optical Orthogonal Codes and Related Combinatorial Designs

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Abstract Using channel polarization technique in optical code-division multiple access, we can spread optical pulses in the spatial domain, in addition to the time and frequency domains. The pattern of transmitting optical pulses in these three dimensions are specified by the codewords of a three-dimensional optical orthogonal codes (3-D OOC). In this work, combinatorial designs related to optimal 3-D OOC are discussed and some constructions of 3-D OOC are given.

Keywords Optical orthogonal codes, constant weight codes, group divisible design, generalized Bhaskar Rao designs.

1 Introduction

Let $\mathbb{I}_n$ be the set $\{0, 1, \ldots, n - 1\}$ and $\mathbb{Z}_n$ be the ring of residues $\mathbb{Z}/(n\mathbb{Z})$. For positive integers $S$, $W$ and $T$, a codeword of a three-dimensional optical orthogonal code (3-D OOC) is defined as a mapping $X$ from $\mathbb{I}_S \times \mathbb{I}_W \times \mathbb{Z}_T$ to \{0, 1\}. We can represent a codeword in two ways. In the first representation, a codeword is identified with a 3-D $S \times W \times T$ matrix $[X(s, w, t)]$, where $X(s, w, t)$ is the image of $(s, w, t) \in \mathbb{I}_S \times \mathbb{I}_W \times \mathbb{Z}_T$ under the mapping $X$. Alternately, a codeword can be represent by the support of the mapping, defined as the set

$$\{(s, w, t) \in \mathbb{I}_S \times \mathbb{I}_W \times \mathbb{Z}_T : X(s, w, t) = 1\}.$$

We define the Hamming correlation function of two codewords $X$ and $Y$ by

$$H_{X,Y}(\tau) := \sum_{s=0}^{S-1} \sum_{w=0}^{W-1} \sum_{t=0}^{T-1} X(s, w, t)Y(s, w, t \oplus \tau),$$

(1)

where “$\oplus$” denotes modulo-$T$ addition, and $\tau$ is an integer between 0 and $T-1$. When $X = Y$ and $\tau = 0$, $H_{X,X}(0)$ is the size of the support of $X$, and is called the Hamming weight of $X$.
An \((S \times W \times T; \omega, \lambda_a, \lambda_c)\) 3-D OOC, denoted by \(C\), is a collection of codewords satisfying following conditions [11]:

1) Hamming weight. \(H_{X;X}(0) = \omega\), for all \(X \in C\),

2) Auto-correlation. \(H_{X;X}(\tau) \leq \lambda_a\), for all \(X \in C\) and \(1 \leq \tau \leq T - 1\),

3) Cross-correlation. \(H_{X,Y}(\tau) \leq \lambda_c\), for all \(X, Y \in C\), \(X \neq Y\), and \(0 \leq \tau \leq T - 1\).

3-D OOC can be applied in optical code-division multiple access (OCDMA) in which optical pulses are spread in three dimensions, namely spatial, frequency and time. The spatial channels come from polarization of the optical channel. The spreading over frequency is achieved by transmitting optical pulses with different wavelengths. The spreading over time means transmitting optical pulses in different time slots, which are also known as time chips. If codeword \(X\) is assigned to a user, then this user transmits an optical pulse at spatial channel \(s\), wavelength \(w\), and time chip \(t\) if and only if \(X(s, w; t \mod T) = 1\). When \(S = W = 1\), the notion of 3-D OOC reduces to the original optical orthogonal code proposed by Chung et al. [3].

Experimental evaluation of OCDMA with spreading in spatial, frequency, and time domains can be found in [7, 14].

Another application of 3-D OOC is digital watermarking for video signal [15]. In this application, we regard a codeword as a 3-D matrix. The cross-sections of the 3-D matrix in the time axis are associated with the frames of picture in a video. A pixel is marked if and only if the corresponding entry in the 3-D matrix is 1.

Example 1 The followings eight 3-D matrices form a \((2 \times 2 \times 2, 0, 1)\) 3-D OOC:

\[
\begin{align*}
\text{s = 0 :} & \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\
\text{s = 1 :} & \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.
\end{align*}
\]

The first (resp. second) row are the first (resp. second) spatial dimension of the 3-D matrices. In each 2-D matrix, the two rows correspond to the two wavelengths, and the two columns correspond to the two time chips. The supports of the codewords are

\[
\begin{align*}
\{(0, 0, 0), (1, 0, 0)\}, & \quad \{(0, 0, 0), (1, 1, 0)\}, \\
\{(0, 0, 0), (1, 1, 1)\}, & \quad \{(0, 0, 0), (1, 1, 1)\}, \\
\{(0, 1, 0), (1, 0, 0)\}, & \quad \{(0, 1, 0), (1, 1, 0)\}, \\
\{(0, 1, 0), (1, 1, 0)\}. & \quad \{(0, 1, 0), (1, 1, 1)\}.
\end{align*}
\]

A wavelength/time plane in a 3-D matrix is called a spatial plane. A 3-D at-most-one-pulse-per-plane code (AMOPPC) is a 3-D OOC if the every spatial plane in every codeword contains at most one optical pulse, i.e.,

\[
\sum_{w=0}^{W-1} \sum_{t=0}^{T-1} X(s, w, t) \leq 1
\]

for all codewords \(X\) and \(s \in \mathbb{Z}_2\). If there is exactly one “1” in every spatial plane in every codeword of a 3-D OOC, then it is called a single-pulse-per-plane code (SPPC). If there are more than one pulses per spatial plane in any codeword, then the 3-D OOC
Ref. | Parameters $S, W, T$ | $\omega$ | $\lambda$ | Code size | Type
--- | --- | --- | --- | --- | ---
[11] | Prime factors of $T$ are all larger than or equal to $SW$ | $SW$ | 1 | $T$ | MPPC
[11] | Prime factors of $W$ and $T$ are all larger than or equal to $S$, $S = p, W = p^2 - 1, T = p, p$ prime. | $S$ | 1 | $W^2T$ | SPPC
[16] | $S = W = p^2 - 1, T = p(p^2 - 1)$ | $p^2 - 1$ | 1 | $p(p^2 - 1)$ | SPTC
[16] | $S = W = p, T = p^2 - 1$ | $p^2 - 1$ | 1 | $p^2$ | SPTC
[12] | $S = W = T = p$, $p$ prime, $1 \leq r \leq p - 2$ | $S$ | $r$ | $W^{\lambda + 1}T^\lambda$ | SPPC
[12] | $S = 4, W = q, T \geq 2$, $q$ is a prime power $\geq 4$, $S = q + 1, W = q, T = p$, $p$ is a prime $> q$. | $S$ | 2 | $W^2T^2$ | SPPC
[12] | $S = 4, W$ is even when $T$ is even. | 3 | 1 | $W^2T$ | SPPC
[18] | $S(S - 1)WT \equiv 0 \mod 2, S \equiv 0, 1 \mod 4$ when $T \equiv 2 \mod 4$ and $W \equiv 1 \mod 2$. | 3 | 1 | $\left(\frac{S}{2}\right)W^2T$ | AMOPPC
[18] | $S = 4, W$ and $T$ satisfy the conditions in Construction 3 | 4 | 1 | $\left(\frac{S}{2}\right)W^2T$ | AMOPPC
This paper | $S, W, T$ satisfy the conditions in Construction 3 | 4 | 1 | $W^2T$ | SPPC
This paper | $S - 1 \leq$ all prime power factors of $W$, $S \leq$ all prime factors of $T$ | 4 | 1 | $W^2T$ | SPPC
This paper | $S = 4, W$ and $T$ satisfy the conditions in Construction 5 and 6 | 4 | 1 | $W^2T$ | SPPC

Table 1: Summary of constructions for 3-D OOC.

is called a multiple-pulse-per-plane code (MPPC). Likewise, a 3-D at-most-one-pulse-per-time code (AMOPPC) is a 3-D OOC satisfying

$$\sum_{s=0}^{S-1} \sum_{w=0}^{W-1} X(s, w, t) \leq 1$$

for all codewords $X$ and $t \in \mathbb{Z}_T$. If equality holds in the above inequality for all codewords and all $t$, then we have a single-pulse-per-time code (SPTC). Some explicit constructions for 3-D OOC in [11, 12, 16, 18] are summarized in Table 1.

For fixed Hamming weight, Hamming auto- and cross-correlation requirement, we want to construct 3-D OOC with large number of codewords. A Johnson-type bound on the number of codewords for general 3-D OOC, $C$, is given in [16],

$$|C| \leq \left[\begin{array}{c|c|c|c|c} \frac{SW}{\omega} & \frac{SWT - 1}{\omega - 1} & \frac{SWT - 2}{\omega - 2} & \cdots & \frac{SWT - \lambda}{\omega - \lambda} \end{array}\right].$$

Within the class of AMOPPC, the we have a tighter bound on code size [18]

$$|C| \leq \left[\begin{array}{c|c|c|c} \frac{SW}{\omega} & \frac{(S - 1)WT}{\omega - 1} & \frac{(S - 2)WT}{\omega - 2} & \cdots & \frac{(S - \lambda)WT}{\omega - \lambda} \end{array}\right].$$

If we remove all the floor operators in the above inequality, we get

$$|C| \leq \left(\frac{\lambda + 1}{\lambda + 1}\right)W^\lambda 1^\lambda.$$

(2)
A 3-D AMOPPC with code size attaining equality in (2) is said to be perfect.

In this work we focus on perfect 3-D SPPC and AMOPPC with $\lambda = 1$, and related combinatorial designs. AMOPPC has the property that the Hamming auto-correlation is identically zero for all time shifts $\tau$. Also, SPPC has the property that the Hamming weight $w$ is equal to the number of spatial channels $S$. Hence, we simplify the notation by referring to an $(S \times W \times T, S, 0, \lambda)$ 3-D SPPC by $(S \times W \times T, \lambda)$-SPPC, and $(S \times W \times T, \omega, 0, \lambda)$ 3-D AMOPPC by $(S \times W \times T, \omega, \lambda)$-AMOPPC.

We study 3-D AMOPPC from a design-theoretical viewpoint, by considering the supports of the codewords as the base blocks of some combinatorial designs. As the spectrum of parameters of perfect 3-D AMOPPC of weight 3 is completely characterized in [18], we will emphasize on 3-D AMOPPC of weight 4 or more. Combinatorial constructions of 2-D OOC and 1-D OOC can be found in [4, 13, 19–22], and the references therein. The parameters of the new constructions in this paper are listed in the last three rows in Table 1.

2 Combinatorial Designs

We review some definitions and results from combinatorial design theory, in particular group divisible design and generalized Bhasker Rao design, which are useful in the construction of 3-D OOC.

Let $v$ be a positive integer and $K$ be a set of positive integers. A group divisible design (GDD) of order $v$ and block sizes from $K$ is a triple $(V, G, B)$, where
1. $V$ is a set of size $v$, whose elements are called points,
2. $G$ is a partition of $V$ into disjoint sets, called groups, and
3. $B$ is a collection of subsets in $V$, called blocks, such that
   (a) each block in $B$ has size in $K$,
   (b) each block intersects every group in $G$ in at most one point, and
   (c) any pair of points from two distinct groups is contained in exactly $\lambda$ blocks of $B$.

We use the notation $GDD(\lambda; v)$ for a group divisible design. The type of the multiset $\{|G| : G \in G\}$ is usually written in an “exponential” notation $i_11^2i_22^3i_33^4\cdots$, which means that there are precisely $i_j$ groups in $G$ with size $j$, for $j = 1, 2, 3, \ldots$. If all groups have the same size $m$, the GDD is said to be uniform, and the corresponding type can be written as $m^s$ for some $s$. When all blocks in a GDD have the same size $k$, we write $GDD(k; v)$ instead. A transversal design is a uniform $GDD(k; v)$ in which every block intersects every group. A transversal design of type $m^s$ is denoted by $TD(k, m)$, where $k$ denotes the block size and $m$ denotes the group size. When every group in $G$ is a singleton, a $GDD(k; v)$ is called a pairwise balanced design (PBD). We use the notation $PBD(k; v)$ for PBD with block sizes in $K$. If all blocks in a pairwise balanced design have the same size, we have a balanced incomplete block design (BIBD). When $\lambda = 1$, the subscript $\lambda$ will be omitted.

Example 2 Let $V = \{1, 2, 3, 4, 5\}$ be the point set, and $G = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ the group set. Let $B$ be the collection
$$B = \{\{1, 2, 4\}, \{1, 3, 5\}, \{2, 5\}, \{3, 4\}\}.$$ Then $(V, G, B)$ is a $GDD(\{2, 3\}; 5)$ of type $1^12^2$. 
Example 3 The following is an example of PBD(\{3, 4\}; 10) from [1, Example I.6.7]. The point set \( \mathcal{V} \) is \( \{0, 1, \ldots, 9\} \) and the blocks are
\[
(1, 4, 9), \ (1, 5, 8), \ (1, 6, 7), \ (2, 4, 7), \ (2, 5, 9), \ (2, 6, 8), \ (3, 4, 8), \ (3, 5, 7), \ (3, 6, 9), \ (0, 1, 2, 3), \ (0, 4, 5, 6), \ (0, 7, 8, 9).
\]

Let \( G \) be a finite abelian group, and let “\( \infty \)” be a special symbol not in \( G \). A generalized Bhaskar Rao design [5] is an \( n \times b \) array with entries in \( G \cup \{\infty\} \), such that
1) each row has exactly \( r \) entries in \( G \),
2) each column contains exactly \( k \) entries in \( G \), and
3) for each pair of distinct rows \((x_1, x_2, \ldots, x_b)\) and \((y_1, y_2, \ldots, y_b)\), the list
\[
x_1 - y_1 : i = 1, 2, \ldots, b, \ x_i \neq \infty \neq y_i,
\]
contains exactly \( \lambda/|G| \) copies of each element in \( G \). We denote a generalized Bhaskar Rao design by \((n, k, \lambda; G)\)-GBRD. It is straightforward to see that (i) \( \lambda \) is a multiple of \(|G|\), (ii) \( bk = nr \), and (iii) \( r(k - 1) = \lambda(n - 1) \). Hence, the number of columns of an \((n, k, \lambda; G)\)-GBRD can be calculated by
\[
b = \frac{n(n - 1)}{k(k - 1)}.
\]

When \( n = k \) and \( \lambda = |G| \) for a cyclic group \( G \), the GBRD is also known as cyclic difference matrix over \( G \). We will denote a cyclic difference matrix over a cyclic group \( G \) by \((n, |G|)\)-CDM.

Example 4 The following \( 4 \times 12 \) array,
\[
\begin{array}{ccccccccccccc}
\infty & 0 & 0 & 1 & \infty & 1 & 1 & 0 & \infty & 3 & 3 & 0 \\
0 & \infty & 0 & 0 & 0 & \infty & 3 & 3 & 0 & \infty & 1 & 1 \\
0 & 1 & \infty & 1 & 2 & 0 & \infty & 2 & 4 & 0 & \infty & 4 \\
0 & 0 & 1 & \infty & 4 & 3 & 0 & \infty & 2 & 1 & 0 & \infty \\
\end{array}
\]
is a \((4, 3, 6; \mathbb{Z}_4)\)-GBRD.

We note that if we replace the special symbol \( \infty \) in a GBRD by 0 and the group elements by 1, then the resulting matrix is the incidence matrix of a BIBD. The GBRD is said to be obtained from the corresponding BIBD by “signing” the incidence matrix by the elements of group \( G \). If we start from a group divisible design and sign the associated incidence matrix by the elements of an abelian group, satisfying similar properties as in GBRD, then the resulting matrix is called a generalized Bhaskar Rao group divisible design (GBRGDD) [9]. In the sequel we are interested in signing uniform GDD. The GBRGDD obtained by signing a GDD\(_s\)(\( K; ms \)) of type \( m^s \) over a finite abelian group \( G \) is denoted by \((K, \lambda; G)\)-GBRGDD of type \( m^s \). If \( K = \{k\} \), the GBRGDD has \( ms \) rows and \( \frac{\lambda(s - 1)}{k(s - 1)}m^2|G| \) columns. We use the notation \((k, \lambda; G)\)-GBRGDD of type \( m^s \) when \( K \) is the singleton \( \{k\} \).

The notion of GBRGDD naturally encompasses the definitions of GBRD and GDD. When the abelian group \( G \) is the trivial group consisting of a single element, then GBRGDD reduces to a GDD. When the type of GBRGDD is \( 1^s \) for some \( s \), it degenerates to a GBRD.

We record some known results below.
Theorem 1 [1, 8.5.c] If the prime power factorization of $m$ is
\[ m = p_1^{e_1} p_2^{e_2} \cdots p_x^{e_x} \]
for some distinct primes $p_1, p_2, \ldots, p_x$, then there exists a $\text{TD}(k, m)$ with
\[ k = \min\{p_1^{e_1}, p_2^{e_2}, \ldots, p_x^{e_x}\} + 1. \]

The next theorem characterizes the existence of uniform GDD of block size four.

Theorem 2 [2] Let $m$ and $s$ be positive integers. A necessary and sufficient condition for the existence of uniform GDD $\lambda(4; ms)$ of type $ms$ is that the design is not $\text{TD}(4, 2)$ and not $\text{TD}(4, 6)$, and that
\[
\lambda(s - 1)m \equiv 0 \mod 3, \\
\lambda s(s - 1)m^2 \equiv 0 \mod 12, \text{ and } s \geq 4.
\]

For a positive integer $n$ with prime factorization $n = p_1 p_2 p_3 \cdots p_e$, an elementary abelian group of order $n$ is defined as the direct product of $\mathbb{Z}_{p_i}$ for $i = 1, 2, \ldots, e$, and is denoted by $EA(n)$. Conditions for the existence of GBRD of block size four signed over elementary abelian group is given in the next theorem.

Theorem 3 [9, Thm 4.13] A necessary condition for a $(4, 4; EA(n))$-GBRD to exist is that $n$ divides $\lambda$. If $n$ divides $\lambda$, then a $(4, 4; EA(n))$-GBRD exists unless
1. $n \equiv \lambda \equiv 2 \mod 4$ when $n$ is even, \\
2. $n = \lambda = 3$ when $n$ is odd.

The next theorem gives some sufficient conditions for the existence of cyclic difference matrix.

Theorem 4 [6, 8] Let $n \geq 5$ be an odd integer. There exists a $(4, n)$-CDM when
1. $\gcd(n, 27) \neq 9$, or \\
2. $n = 9i$ for $5 \leq i \leq 31$.

By adapting Wilson’s fundamental construction of pairwise balanced design [23], we have the following product construction of GBRGDD, which is in fact a special case of a more general theorem in [9]. Nevertheless, this special case is enough for the subsequent constructions.

Theorem 5 [9, Thm 5.2] Let $H$ be a subgroup of a finite abelian group $G$. Suppose that is a $(K_1, \lambda_1; G/H)$-GBRGDD of type $m_1^{k_1}$, and a $(K_2, \lambda_2; H)$-GBRGDD of type $m_2^{k_2}$ for all $k_1 \in K_1$. Then there is a $(K_2, \lambda_1 \lambda_2; G)$-GBRGDD of type $(m_1 m_2)^s$.
Using Theorem 5, we can construct cyclic difference matrix recursively. Recall that a \((k,n,Z_n)\)-GBRGDD of type 1 is the same as a \((k,s)\)-CDM. When applied to this special case, Theorem 5 says that from two cyclic difference matrices \((k,r)\)-CDM over \(Z_r\) and \((k,s)\)-CDM over \(Z_s\), we can construct a \((k,rs)\)-CDM over \(Z_{rs}\). The idea is essentially the same as the Kronecker-product construction in [17].

**Theorem 6** [17] If the prime factorization of \(T\) is \(p_1 p_2 \cdots p_e\) for (not necessarily distinct) primes \(p_1, p_2, \ldots, p_e\), then there exists a \((k,T)\)-CDM for \(k = \min\{p_1, p_2, \ldots, p_e\}\).

**Proof** For \(i = 1, 2, \ldots, e\), we have cyclic difference matrix \((k,p_i)\)-CDM, constructed from the first \(k\) rows of the mod \(p_i\) multiplication table for example. By combining them using the product construction in Theorem 5, we have a \((k,T)\)-CDM. \(\square\)

### 3 Combinatorial Constructions of Perfect AMOPPC and SPPC

The main result of this work is the following connection between perfect 3-D AMOPPC and GBRGDD signed over cyclic group.

**Theorem 7** The followings are equivalent:
1. A perfect \((S \times W \times T, \omega, 1)\)-AMOPPC,
2. \((\omega, T; Z_T)\)-GBRGDD of type \(W^S\).

By “equivalent” we mean that given a perfect \((S \times W \times T, \omega, 1)\)-AMOPPC we can construct an \((\omega, T; Z_T)\)-GBRGDD of type \(W^S\), and vice versa.

We omit the proof of Theorem 7, but give an example instead.

**Example 5** The GBRGDD associated with the 3-D SPPC in Example 1 is the following \((4,2;Z_2)\)-GBRGDD of type \(2^2\):

\[
\begin{array}{cccccc}
0 & 0 & 0 & \infty & \infty & \infty \\
0 & \infty & \infty & \infty & 0 & 0 \\
0 & 0 & \infty & \infty & 0 & \infty \\
0 & \infty & \infty & \infty & 0 & 0
\end{array}
\]

The first two rows correspond to the first group, while the last two rows correspond to the second. Each column is associated with a codeword of the 3-D OOC.

Utilizing the equivalence between GBRGDD and perfect AMOPPC, we have the following product construction from Theorem 5.

**Construction 1** Given a perfect \((S \times W_1 \times T_1, k, 1)\)-AMOPPC and a perfect \((k \times W_2 \times T_2, \omega, 1)\)-AMOPPC, we can construct a perfect \((S \times W_1 W_2 \times T_1 T_2, \omega, 1)\)-AMOPPC.

Given a GBRD signed over the direct sum of two abelian groups, we can construct perfect 3-D AMOPPC as well.

**Theorem 8** Let \(G = H \times Z_T\) be the direct product of an abelian group \(H\) of size \(W\) and a cyclic group \(Z_T\) of size \(T\). If there exists an \((S, \omega, WT; G)\)-GBRD, then there exists a perfect \((S \times W \times T, \omega, 1)\)-AMOPPC.
Proof (sketch) Let \( M \) be an \((S, \omega, W, T; H \times Z_T)\)-GBRD. The size of \( M \) is \( S \times W \times S(S−1)/(\omega(\omega − 1)) \). Each a column of \( M \) has \( \omega \) entries in group \( G \). Consider a particular column. The rows of the \( \omega \) entries specify the spatial channels. Let the content the \( \omega \) entries in \( G \) be \((w_i, t_i) : i = 1, 2, \ldots, \omega\) for \( x \) running over the elements in \( H \). Repeating the above procedure for each columns of \( M \), we obtain \( W^2TS(S−1)/(\omega(\omega − 1)) \) codewords, which form a perfect \((S \times W \times T, \omega, 1)\)-AMOPPC. □

In Theorem 5, if the first GBRGDD is a GDD(\( \omega, WS \)) of type \( WS \) (by taking \( K_1 = \{\omega\}, G = H \) so that \( G/H \) is the trivial group of size 1, \( \lambda_1 = 1 \), and \( m_1 = W \)), and the second GBRGDD is an \((\omega, T)\)-CDM (by taking \( K_2 = \{\omega\}, \lambda_2 = T, H = Z_T, \) and \( m_2 = 1 \)), we obtain

**Construction 2** If there exists a GDD(\( \omega, WS \)) of type \( WS \) and an \((\omega, T)\)-CDM, then there exists a perfect \((S \times W \times T, \omega, 1)\)-AMOPPC.

Combining Theorems 2 and 4 and Construction 2, we get

**Construction 3** Let \( S \) and \( W \) be positive integers satisfying

1. \((S − 1)W \equiv 0 \mod 3\),
2. \( S(S−1)W^2 \equiv 0 \mod 12\),
3. \( S \geq 4 \), and
4. \((4, 2) \neq (S, W) \neq (4, 6),\)

and \( T \geq 5 \) be odd integer satisfying

1. \( \gcd(T, 27) \neq 9 \), or
2. \( T = 9i \) for \( 5 \leq i \leq 31 \),

then there exists a perfect \((S \times W \times T, 4, 1)\)-AMOPPC.

We can improve the parameters in the construction of SPPC by Kim et al. in [11].

**Construction 4** Let the prime power factorization of \( W \) be \( W = q_1q_2\cdots q_x \), and the prime factorization of \( T \) be \( T = p_1p_2\cdots p_y \) (\( q_1, q_2, \ldots, q_x \) are distinct prime powers and \( p_1, p_2, \ldots, p_y \) are not necessarily distinct primes). If

\[
S \leq \min\{q_1 + 1, q_2 + 1, \ldots, q_x + 1, p_1, p_2, \ldots, p_y\},
\]

then there exists a perfect \((S \times W \times T, 1)\)-SPPC.

**Proof** Since \( S \leq q_i + 1 \) for all \( i \), we can construct a transversal design \( TD(S, W) \) by Theorem 1. Since \( S \leq p_j \) for all \( j \), we have a cyclic difference matrix \((S, T)\)-CDM by Theorem 6. By Construction 2, we obtain a perfect \((S \times W \times T, 1)\)-SPPC.

### 4 Single-pulse-per-plane code with \( S = 4 \)

In this section, we collect some constructions for the special case of SPPC with four spatial channels.
Example 6 From the following $(4, 4, 4; Z_2 \times Z_2)$-GBRD taken from [10]

\[
\begin{bmatrix}
00 & 00 & 00 & 00 \\ 00 & 01 & 10 & 11 \\ 00 & 10 & 11 & 01 \\ 00 & 11 & 01 & 10
\end{bmatrix}
\]

we can construct a $(4 \times 4)$-$\text{SPPC}$ by Theorem 8. The supports of the eight codewords are

\[
\{(1, 0, 0), (2, 0, 0), (3, 0, 0), (4, 0, 0)\}, \{(1, 1, 0), (2, 1, 0), (3, 1, 0), (4, 1, 0)\},
\{(1, 0, 0), (2, 1, 0), (3, 1, 0), (4, 1, 0)\}, \{(1, 1, 0), (2, 1, 1), (3, 0, 0), (4, 0, 1)\},
\{(1, 0, 0), (2, 1, 0), (3, 1, 1), (4, 0, 1)\}, \{(1, 1, 0), (2, 0, 0), (3, 0, 1), (4, 1, 1)\},
\{(1, 0, 0), (2, 1, 1), (3, 0, 1), (4, 1, 0)\}, \{(1, 1, 0), (2, 0, 1), (3, 1, 1), (4, 0, 0)\}.
\]

Example 7 From the $(4, 4, 8; Z_2 \times Z_4)$-GBRD from [10]

\[
\begin{bmatrix}
00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 \\ 00 & 01 & 02 & 03 & 10 & 11 & 12 & 13 \\ 00 & 03 & 13 & 12 & 11 & 10 & 02 & 01 \\ 00 & 12 & 03 & 11 & 13 & 01 & 10 & 02
\end{bmatrix}
\]

we can construct a $(4 \times 2 \times 4, 1)$-$\text{SPPC}$ and a $(4 \times 4, 1)$-$\text{SPPC}$ by Theorem 8.

More generally we have the following construction for even $T$.

Construction 5

1. For $T = 2$, there exists a perfect $(4 \times W \times 2, 1)$-$\text{SPPC}$ for all even $W$.
2. For even $T \geq 4$, there exists a perfect $(4 \times W \times T, 1)$-$\text{SPPC}$ if $W$ is even and $WT \neq 2^e 3$ for any exponent $e$.

Proof The first part of the construction $T = 2$ follows from Theorem 3 and Construction 8.

For the second part, let $W = 2^i j$ and $T = 2^2 j$, where $i$ and $j$ are odd integers. By assumption, $ij \neq 3$. From Theorem 3 and Construction 1, we have a $(4 \times i \times j, 1)$-$\text{SPPC}$. By combining several copies of the SPPC in Examples 6 and 7, and applying Construction 1 again, we have a $(4 \times W \times T, 1)$-$\text{SPPC}$.

The last construction is for odd $T$.

Construction 6

1. For $T = 3$, there exists a perfect $(4 \times W \times 3, 1)$-$\text{SPPC}$ for all $W \neq 2$ mod 4.
2. For odd $T \geq 5$ satisfying $\gcd(T, 27) = 9$ or $T = 9i$ for $5 \leq i \leq 31$, there exists a perfect $(4 \times W \times T, 1)$-$\text{SPPC}$ for $3 \leq W \neq 6$.

Proof Suppose $T = 3$ and $W \neq 2$ mod 4. We have a $(4, 4, 3W; EA(W) \times Z_3)$-GBRD by Theorem 3. We can then construct a perfect $(4 \times W \times 3, 1)$-$\text{SPPC}$ by Theorem 8.

The second part follows directly from Construction 3 by setting $S = 4$.  \[\square\]
References

7. Garg, M.: Performance analysis and implementation of three dimensional codes in optical code division multiple access system. Master, Thapar University, Punjab, India (2012)